

NEW EXAMPLES OF KOCHEN-SPECKER TYPE CONFIGURATIONS ON THREE QUBITS

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ABSTRACT. A new example of a critical Kochen-Specker type configuration of 36 rays in 8-dimensional space (the Hilbert space of a triple of qubits) is constructed. This configuration is shown to admit an extension up to a saturated Kochen-Specker type configuration containing 64 rays. A natural multicoloured generalization of the Kochen-Specker theory is given based on a concept of entropy of a saturated configuration of rays.

1. INTRODUCTION

In this paper I describe a new family of Kochen-Specker type configurations of pure states of a finite dimensional quantum system. Such configurations are sometimes termed “non-colourable” configurations [1], or also Bell-Kochen-Specker configurations by some authors [2, 3, 4] to stress the fact that the original Kochen-Specker construction [5] in three dimensions can be seen as a development of ideas about Bell’s inequalities in quantum mechanics [6].

The Kochen-Specker type configurations are of interest for the foundations of quantum mechanics since they do not refer directly to the concept of *probability*. An additional motivation to study such configurations is, on one hand, due to the links to some issues of quantum gravity [7, 8, 9, 10, 11, 12], and, on the other hand, due to the links to quantum computers.

Usually one tries to find a Kochen-Specker type configuration which contains as less rays as possible. For example, there exists a example of a Kochen-Specker type configuration in four dimensions consisting of just 18 rays [2] which has been tested experimentally. The dimension four can be perceived as the dimension a pair of qubits ($4 = 2 \times 2$). In the present paper the dimension is eight, which is the dimension of a three-qubit system ($8 = 2 \times 2 \times 2$). Perhaps the most interesting result obtained in the present paper is a new *saturated* Kochen-Specker type configuration on a triple of qubits which contains a new *critical* Kochen-Specker type configuration of 36 rays. It would be of interest to test these configurations experimentally.

This year IBM has announced [13] a big step forward towards a creation of a full scale quantum computer. Loosely speaking, what they did is put three superconducting qubits on one chip. This triple of qubits is expected to become a basic building block of a scalable quantum computer. In this context the Kochen-Specker type configurations in eight dimensions become of particular interest.

A triple of qubits is a rather special quantum object on its own. To observe the EPR-type correlations [14] one may work with only two qubits, but to observe the *really* weird features of quantum theory it is better to have at least three qubits. For instance, it has been recently shown [15] that a triple of qubits suffices to refute the conjecture of Peres about quantum nonlocality and entanglement distillability. One can observe Berry's phase [16] on a triple of qubits and describe it in terms of quantum groups [17]. There exists a non-trivial link between the E_8 -root system and the theory of three qubits found in [18]. It turns out that the rays represented by the roots of an E_8 -type Lie algebra yield an example of a *saturated* Kochen-Specker type configuration and that this fact can be generalized in several different ways. One can obtain an infinite family of orthoalgebras [19] starting from the E_8 -root system which can be of interest in the approach to quantum gravity advocated in [10]. On the other hand one can scale up and deform this example on multiple qubits [20], as well as one can consider other root systems [21].

In the present paper I concentrate on the case of a three qubit system, although other important Kochen-Specker examples are known in other dimensions [1, 2, 4, 5, 20, 21, 22, 23, 24, 25, 26]. The paper can be split in two parts. The first part is more abstract and theoretical and the second part is more concrete and applied. In the abstract part I introduce a concept of an *entropy* of a configuration of rays. This naturally leads to a *multicoloured* generalization of the Kochen-Specker theory. In the applied part I describe new Kochen-Specker configurations on three qubits which yield a collection of convenient model examples to test many ideas. First I describe a new *saturated* Kochen-Specker type configuration consisting of 64 rays. Another saturated Kochen-Specker type configuration known in 8-dimensions stems from the E_8 -root system and contains 120 rays [18]. It is an extension of an example due to [27] which contains 40 rays. It turns out that one cannot find an isomorphic copy of this 40-rays configuration in the 64 rays mentioned, so we obtain a new example. A search for critical sub-configurations in this 64-rays configuration yields a critical Kochen-Specker type configuration consisting of 36 rays.

After that I describe a family of sub-configurations in this collection of 64 rays which are critical in another sense: not with respect to the number of rays involved, but with respect to the number of complete measurements involved to illustrate the non-colourability of configuration. This is formalized in a definition of a *tropical* configuration and a concept of *tropical dimension*. It turns out that this leads to another new family of saturated configurations of rays and by that one obtains a new family of examples of orthoalgebras [12, 28, 29].

2. TERMINOLOGY AND NOTATION

Let \mathcal{H} be a finite dimensional Euclidean space over the field of complex numbers \mathbb{C} (the Hilbert space of a finite dimensional quantum system). Denote the dimension $d = \dim_{\mathbb{C}} \mathcal{H}$. Denote the inner product of two vectors $v, w \in \mathcal{H}$ as $\langle v, w \rangle$ and assume the convention that $\langle -, - \rangle$ is linear with respect to the second argument. The null-vector is denoted as $0_{\mathcal{H}} \in \mathcal{H}$, and for $v \neq 0_{\mathcal{H}}$, $\mathbb{C}v$ denotes the ray in \mathcal{H} represented by v . The collection of all rays in \mathcal{H} is denoted as $\mathbb{P}\mathcal{H}$. The norm of a vector v is $\|v\| := \langle v, v \rangle^{1/2}$, and we write $\mathbb{C}v \perp \mathbb{C}w$ iff $\langle v, w \rangle = 0$.

Take a collection $\mathcal{M} = \{x_i\}_{i=0}^{n-1}$ of n rays in \mathcal{H} . For every positive integer $m \in \mathbb{Z}_{>0}$, put

$$\mathcal{P}_{\perp}^{(m)}(\mathcal{M}) := \{U \subseteq \mathcal{M} \mid \#U = m, \text{ and } \forall x, y \in U : y \neq x \Rightarrow x \perp y\},$$

where $\#$ denotes cardinality of a set. Note that $\mathcal{P}_{\perp}^{(m)}(\mathcal{M})$ is empty if $m > d$ or $m > n$. In analogy with this notation, we define $\mathcal{P}_{\not\perp}^{(m)}(\mathcal{M})$ as a collection of all subsets of \mathcal{M} of size m consisting of mutually non-orthogonal rays, $m \in \mathbb{Z}_{>0}$.

Definition 1. A collection $\mathcal{M} = \{x_i\}_{i=0}^{n-1}$ of rays $x_i \in \mathbb{P}\mathcal{H}$, $i = 0, 1, \dots, n-1$, is called KS-colourable (“KS” stands for Kochen-Specker) iff there exists a function $f : \mathcal{M} \rightarrow \{0, 1\}$ such that

$$\forall x, y \in \mathcal{M} : x \perp y \Rightarrow f(x) + f(y) \leq 1,$$

and

$$\forall U \in \mathcal{P}_{\perp}^{(d)}(\mathcal{M}) : \sum_{x \in U} f(x) = 1.$$

The possible values 0 and 1 of the function f are termed colours, and a function f is termed a KS-colouring.

It can happen that a given collection \mathcal{M} admits many KS-colourings, but it can also happen that no KS-colourings exist.

Definition 2. A Kochen-Specker type configuration in \mathcal{H} is a finite collection \mathcal{M} of rays in \mathcal{H} which does not admit a KS-colouring.

To simplify the terminology, I leave out the word ‘type’ in what follows. It is of interest to consider special classes of collections \mathcal{M} .

Definition 3. *A collection of rays \mathcal{M} in \mathcal{H} , is termed saturated iff for every $W \in \cup_{m>0} \mathcal{P}_{\perp}^{(m)}(\mathcal{M})$, there exists $U \in \mathcal{P}_{\perp}^{(d)}(\mathcal{M})$ such that $U \supseteq W$.*

Definition 4. *Let $k \in \{1, 2, \dots, d\}$. A collection of rays \mathcal{M} in \mathcal{H} , $\dim_{\mathbb{C}} \mathcal{H} = d$, is termed k -strict iff*

$$\mathcal{M} = \cup \{U \mid U \in \mathcal{P}_{\perp}^{(k)}(\mathcal{M})\}.$$

Note that any collection \mathcal{M} is 1-strict. Every saturated collection is k -strict for every $k = 1, 2, \dots, d$. The importance of saturated configurations is stressed in [18, 19]. The d -strict configurations naturally appear in [30, 31] where the authors analyse an example of a Kochen-Specker configuration [22] in terms of hyper-graphs.

If \mathcal{H} is the Hilbert space of a finite dimensional quantum system then given a collection of rays \mathcal{M} in \mathcal{H} one may perceive the elements $x \in \mathcal{M}$ as elementary binary measuring devices represented by orthogonal projectors $\hat{\pi}_\psi$, $x = \mathbb{C}\psi$, $\psi \in \mathcal{H}$. In experiment it is of interest to have a collection \mathcal{M} with the smallest number of elements possible. This motivates the following definition:

Definition 5. *A collection \mathcal{M} of rays in \mathcal{H} is called a critical Kochen-Specker configuration iff \mathcal{M} is a Kochen-Specker configuration, while for every $x \in \mathcal{M}$ the collection $\mathcal{M} \setminus \{x\}$ is not a Kochen-Specker configuration.*

Note that a given Kochen-Specker configuration \mathcal{M} may have many critical Kochen-Specker sub-configurations $\mathcal{N} \subseteq \mathcal{M}$ and their sizes do not need to coincide. The smallest known critical Kochen-Specker configuration in four dimensions contains 18 rays [2]. In the present paper I construct a new critical Kochen-Specker configuration of 36 rays in eight dimensions.

Given a collection of rays \mathcal{M} in \mathcal{H} , it is natural to associate to \mathcal{M} an undirected graph $\Gamma_{\mathcal{M}} = (V, E)$ with vertices $V = \mathcal{M}$ and edges $E = \mathcal{P}_{\perp}^{(2)}(\mathcal{M})$, i.e. a pair of vertices $x, y \in V$ is connected with an edge iff $y \perp x$. The elements of $\mathcal{P}_{\perp}^{(m)}(\mathcal{M})$ correspond to *cliques* in $\Gamma_{\mathcal{M}}$ of size $m \in \mathbb{Z}_{>0}$, and the elements of $\mathcal{P}_{\perp}^{(m)}(\mathcal{M})$ correspond to *anticliques* in $\Gamma_{\mathcal{M}}$ of size $m \in \mathbb{Z}_{>0}$. If the configuration \mathcal{M} is saturated, then all maximal cliques have the same size d .

If \mathcal{M} is d -strict then it is natural to associate to it a hyper-graph $\tilde{\Gamma}_{\mathcal{M}} = (V, \tilde{E})$ with vertices $V = \mathcal{M}$ and hyper-edges $\tilde{E} = \mathcal{P}_{\perp}^{(d)}(\mathcal{M})$. To

describe the relation \perp on the set of rays \mathcal{M} , it suffices to describe the collection of hyper-edges \tilde{E} .

A maximal clique $U \in \mathcal{P}_{\perp}^{(d)}(\mathcal{M})$ in the orthogonality graph $\Gamma_{\mathcal{M}}$ corresponds to a *complete* quantum measuring device. Its action consists in measuring simultaneously the elementary binary observables corresponding to $x_0, x_1, \dots, x_{d-1} \in \mathbb{P}\mathcal{H}$, $U = \{x_i\}_{i=0}^{d-1}$. In experiment it is also of interest to minimize the set-up not with respect to the number of rays, but with respect to the number of complete measuring devices involved to observe the Kochen-Specker effect.

Definition 6. Let \mathcal{M} be a Kochen-Specker configuration in a complex Euclidean space \mathcal{H} of dimension d . The smallest size q of a collection of rays $\{x_i\}_{i=0}^{q-1} \subseteq \mathcal{M}$ such that $\mathcal{N} := \{x_0, x_1, \dots, x_{q-1}\}$ does not admit a KS-colouring is called the critical Kochen-Specker dimension of \mathcal{M} . Notation: $\dim_{KS}(\mathcal{M})$.

Note that \mathcal{M} is a critical Kochen-Specker configuration if and only if $\dim_{KS}(\mathcal{M}) = \#\mathcal{M}$. One can generalize this definition as follows.

Definition 7. Let $k \in \{1, 2, \dots, d\}$. Let \mathcal{M} be a k -strict Kochen-Specker configuration in $\mathcal{H} = \mathbb{C}^d$, $\mathcal{M} = \cup\{U \mid U \in \mathcal{P}_{\perp}^{(k)}(\mathcal{M})\}$. The smallest size q of a collection of cliques $\{U_i\}_{i=0}^{q-1} \subseteq \mathcal{P}_{\perp}^{(k)}(\mathcal{M})$, such that $\mathcal{T} := \cup_{i=0}^{q-1} U_i$ does not admit a KS-colouring is called a k -critical dimension of \mathcal{M} . Notation: $\dim_k(\mathcal{M})$.

Note that if \mathcal{M} is a saturated Kochen-Specker configuration then all the dimensions $\dim_k(\mathcal{M})$, $k = 1, 2, \dots, d$, are defined. Loosely speaking, $\dim_1(\mathcal{M}) = \dim_{KS}(\mathcal{M})$ is the smallest number of elementary binary measuring devices necessary to “see” Kochen-Specker on a configuration \mathcal{M} , and $\dim_d(\mathcal{M})$ is the smallest number of complete quantum measurements needed to “see” Kochen-Specker on \mathcal{M} .

Definition 8. Let \mathcal{M} be a k -strict Kochen-Specker configuration of rays in \mathcal{H} , $d = \dim_{\mathbb{C}} \mathcal{H}$, $k \in \{1, 2, \dots, d\}$. The configuration \mathcal{M} is called k -critical iff there exists a collection of cliques $\{U_i\}_{i=0}^{\dim_k(\mathcal{M})-1} \subseteq \mathcal{P}_{\perp}^{(k)}(\mathcal{M})$, such that $\mathcal{M} = \cup_{i=0}^{\dim_k(\mathcal{M})-1} U_i$.

Note that a 1-critical configuration ($k = 1$ in the definition) is just another name for a critical Kochen-Specker configuration. Since the other extreme case $k = d$ is of special interest, it makes sense to introduce a pair of new (more convenient) alias names. A d -critical configuration \mathcal{M} would be also termed a *tropical* Kochen-Specker configuration. The d -critical dimension $\dim_d(\mathcal{M})$ of a d -strict Kochen-Specker

configuration \mathcal{M} would be also termed its *tropical* Kochen-Specker dimension. In the main text of this paper we are going to consider examples of critical and tropical Kochen-Specker configurations on three qubits.

3. ENTROPY OF A CONFIGURATION

Let us first discuss the general motivation [6, 32, 33] behind the definition of a KS-colouring and then consider its possible generalizations.

In classical (i.e. “before quantum”) physics if we deal with observables A, B, C, \dots of a physical system which can take only two values, say 0 and 1, we tend to think about them in terms of Venn diagrams. The observables are certain subsets $A, B, C, \dots \subseteq \Omega$ of a given set Ω . More precisely, to construct a probabilistic model of experiment one needs to define a σ -algebra of subsets \mathcal{F} on Ω and consider a probability measure P on (Ω, \mathcal{F}) . The observables are then identified with measurable subsets $A, B, C, \dots \in \mathcal{F}$, and it is expected that the empirical frequency of an event, for instance, $A = 1$, tends, as we repeat the same experiment many times N , to $P(A)$ as $N \rightarrow \infty$.

At the same time in every particular realization of an experiment we observe a concrete outcome. For example, let $A = 1$, $B = 1$, and $C = 0$. Forget about the repetitions of experiment. What is a reason for this particular outcome? A classical answer would be that what actually takes place in the experiment is a point $\omega \in \Omega$, and in this particular observation we have $\omega \in (A \cap B) \setminus C$. In other words a point ω is an elementary reason why A, B, C, \dots happen or do not happen.

Take now a finite collection $\mathcal{M} = \{A_1, A_2, \dots, A_n\}$ of classical binary observables. Every point $\omega \in \Omega$ of the probability space (Ω, \mathcal{F}, P) of our experiment induces a function $f_\omega : \mathcal{M} \rightarrow \{0, 1\}$ with the following properties:

$$\forall i, j \in \{1, 2, \dots, n\} : A_i \cap A_j = \emptyset \Rightarrow f_\omega(A_i) + f_\omega(A_j) \leq 1,$$

and for every partition $\{A_{i_\alpha}\}_{\alpha=1}^d \subseteq \mathcal{M}$ of the space $\Omega = \sum_{\alpha=1}^d A_{i_\alpha}$, holds:

$$\sum_{\alpha=1}^d f_\omega(A_{i_\alpha}) = 1.$$

These two properties is precisely what the definition of a Kochen-Specker colouring tries to mimic. More precisely, one restricts only to quantum observables represented by 1-dimensional orthogonal projectors $\hat{\pi}_{\psi_1}, \hat{\pi}_{\psi_2}, \dots, \hat{\pi}_{\psi_n}$. The emptiness of an intersection $A_i \cap A_j = \emptyset$ is then translated into orthogonality of rays $\mathbb{C}\psi_i \perp \mathbb{C}\psi_j$. A Kochen-Specker configuration can be called a classically indeterministic object

since one cannot perceive the measurements of the corresponding observables in terms of an underlying space Ω of elementary reasons.

It is nonetheless of interest to consider the following situation. Let \mathcal{M} be a *saturated* collection of rays in d -dimensional Euclidean space \mathcal{H} , which does admit a Kochen-Specker colouring $f : \mathcal{M} \rightarrow \{0, 1\}$. For example, if $\{\psi_i\}_{i=0}^{d-1}$ and $\{\varphi_j\}_{j=0}^{d-1}$ is an *unbiased* pair of orthonormal bases,

$$|\langle \psi_i, \varphi_j \rangle|^2 = 1/d, \quad i, j = 0, 1, \dots, d-1,$$

then the union $\mathcal{M} = \{\mathbb{C}\psi_i\}_{i=0}^{d-1} \cup \{\mathbb{C}\varphi_j\}_{j=0}^{d-1}$ is such a collection. There exists a unique i_0 such that $f(\mathbb{C}\psi_{i_0}) = 1$, and there exists a unique j_0 such that $f(\mathbb{C}\varphi_{j_0}) = 1$. If we look now, say, at a pair of observables represented by the operators

$$\widehat{Q} := \sum_{i=0}^{d-1} i \widehat{\pi}_{\psi_i}, \quad \widehat{P} := \sum_{j=0}^{d-1} j \widehat{\pi}_{\varphi_j},$$

then the illusion is that the colouring f gives us a knowledge of the values of the quantum observables Q and P represented by \widehat{Q} and \widehat{P} , respectively, despite of the fact that $[\widehat{P}, \widehat{Q}] \neq 0$. The value of Q is i_0 , and the value of P is j_0 . Since the bases are unbiased, this is similar [34] to as if we know the coordinate and momentum of a one-dimensional quantum particle with absolute certainty.

These preliminary considerations motivate the following generalization of a Kochen-Specker colouring. Let us look at a function $f : \mathcal{M} \rightarrow [0, 1]$, such that for every $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$ holds:

$$\sum_{x \in U} f(x) = 1.$$

For any saturated configuration \mathcal{M} such a function always exists: one may take $f(x) = 1/d$, for every $x \in \mathcal{M}$. Since $0 \leq f(x) \leq 1$ can be perceived as probability weights, let us define an *entropy*:

$$S_U^f := - \sum_{x \in U} f(x) \log f(x),$$

for every $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$. If f is a Kochen-Specker colouring, then $S_U^f = 0$, for every U . Note that this does not imply that there is a quantum state with such entropies [35, 36]. If $f(x) = 1/d$, for all $x \in \mathcal{M}$, then the entropies are the same $S_U^f = \log(d)$ and their value is maximal. If \mathcal{M} is a Kochen-Specker configuration, then one cannot construct a function f such that $S_U^f = 0$, for all $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$.

Take a saturated collection of rays \mathcal{M} and denote by $\mathcal{K}(\mathcal{M})$ the collection of functions $f : \mathcal{M} \rightarrow [0, 1]$ such that $\sum_{x \in U} f(x) = 1$ and

$S_U^f = S_{U'}^f$, for all $U, U' \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$. Then one may write S_*^f instead of S_U^f , without specifying the lower index. Note that we already know that the class $\mathcal{K}(\mathcal{M})$ is not empty, and that S_*^f interpolates between 0 and $\log(d)$ as f varies over $\mathcal{K}(\mathcal{M})$.

Definition 9. Let \mathcal{M} be a saturated collection of rays in a Euclidean space \mathcal{H} of dimension d . The real number

$$S(\mathcal{M}) := \inf\{S_*^f \mid f \in \mathcal{K}(\mathcal{M})\}, \quad (1)$$

is called an entropy of \mathcal{M} .

Note that the statistical weight

$$D := \exp(S(\mathcal{M}))$$

corresponding to the entropy (1) can be perceived as a new dimension describing $\mathcal{M} \subseteq \mathbb{P}\mathcal{H}$, $1 \leq D \leq d$, $d = \dim_{\mathbb{C}} \mathcal{H}$.

Using this new definition one can observe that if \mathcal{M} admits a Kochen-Specker colouring, then the entropy $S(\mathcal{M}) = 0$. One may speculate that a semiclassical limit over a family of saturated Kochen-Specker configurations $\{\mathcal{M}_i\}_{i=0}^\infty$ would correspond to $S(\mathcal{M}_i) \rightarrow 0$, as $i \rightarrow \infty$.

4. MULTICOLOURED KOCHEN-SPECKER

Let \mathcal{M} be a saturated collection of rays in Euclidean space \mathcal{H} of dimension d . Let $\mathcal{K}(\mathcal{M})$ denote the class of functions $f : \mathcal{M} \rightarrow [0, 1]$ described in the previous section. Let us restrict this class of functions as follows. Assume that the range of values of f is given by $w_1, w_2, \dots, w_s \in]0, 1[$, where $s \leq d$, and $w_\alpha < w_\beta$, if $\alpha < \beta$. Denote

$$N_\alpha^U = \#\{x \in U \mid f(x) = w_\alpha\},$$

where $\alpha = 1, 2, \dots, s$. Then we have:

$$\sum_{\alpha=1}^s N_\alpha^U = d, \quad \sum_{\alpha=1}^s N_\alpha^U w_\alpha = 1, \quad \sum_{\alpha=1}^s N_\alpha^U w_\alpha \log w_\alpha = -S_*^f, \quad (2)$$

where the notation S_*^f is same as in the previous section. Consider *rational* weights $w_\alpha \in \mathbb{Q}$, $\alpha = 1, 2, \dots, s$. Then there exists a positive integer Γ such that

$$w_\alpha = q_\alpha / \Gamma,$$

where q_α are positive integers, $\alpha = 1, 2, \dots, s$. We assume that Γ is chosen as small as possible. We obtain:

$$\sum_{\alpha=1}^s N_\alpha^U = d, \quad \sum_{\alpha=1}^s N_\alpha^U q_\alpha = \Gamma, \quad \sum_{\alpha=1}^s N_\alpha^U q_\alpha \log q_\alpha = \Gamma(-S_*^f + \log \Gamma),$$

for every $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$. Factorize each q_α , $\alpha = 1, 2, \dots, s$, into primes:

$$q_\alpha = p_1^{m_{\alpha,1}} p_2^{m_{\alpha,2}} \dots p_k^{m_{\alpha,k}},$$

where $p_1 < p_2 < \dots < p_k$ are prime numbers, $m_{\alpha,l}$ are non-negative integers, $l = 1, 2, \dots, k$, and we assume that for every l there exists α such that $m_{\alpha,l} > 0$. Then the third equation yields:

$$\sum_{\alpha=1}^s N_\alpha^U q_\alpha \sum_{l=1}^k m_{\alpha,l} \log p_l = \Gamma(-S_*^f + \log \Gamma),$$

If $p_1 > 1$ then the collection of logarithms $\{\log p_l\}_{l=1}^k$ is linearly independent over \mathbb{Z} , and for every $l = 1, 2, \dots, k$, and every $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$, we obtain:

$$\sum_{\alpha=1}^s N_\alpha^U q_\alpha m_{\alpha,l} = K_l,$$

where K_l are some integers, $l = 1, 2, \dots, k$. In other words, the left-hand side of the latter equation, just like the left-hand sides of $\sum_{\alpha=1}^s N_\alpha^U = d$ and $\sum_{\alpha=1}^s N_\alpha^U q_\alpha = \Gamma$, should not depend on the choice of $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$. If $p_1 = 1$ then one obtains just the same equations, but only for $l = 2, 3, \dots, k$.

Definition 10. A function $f : \mathcal{M} \rightarrow \mathbb{Q} \cap]0, 1[$ such that $f \in \mathcal{K}(\mathcal{M})$ is called a rational colouring of a saturated configuration of rays \mathcal{M} . The prime numbers p_1, p_2, \dots, p_k associated with f as described above are termed the prime colours of f , and the integers q_1, q_2, \dots, q_s associated with f are termed the mixed colours of f .

Note, that if all integers q_1, q_2, \dots, q_s are prime, then we conclude that the value N_α^U must be the same for every $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$. In this case one may write:

$$N_\alpha^U = N_\alpha,$$

for $\alpha = 1, 2, \dots, s$, $N_\alpha \in \mathbb{Z}_{>0}$. The second equation in (2) reduces just to $\sum_{\alpha=1}^s N_\alpha p_\alpha = \Gamma$ and can be perceived as a way to recover Γ .

Definition 11. Let \mathcal{M} be a collection of rays on \mathcal{H} , $\dim_{\mathbb{C}} \mathcal{H} = d$. Let $d = N_0 + N_1 + \dots + N_{s-1}$ be a partition of d into a sum of $s \geq 2$ positive integers, where $N_\alpha \geq N_\beta$, if $\alpha < \beta$. A function $f : \mathcal{M} \rightarrow \{0, 1, \dots, s-1\}$ is termed a colouring of \mathcal{M} compatible with this partition iff for every clique $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$ and for every $\alpha = 0, 1, \dots, s-1$, holds: $\#\{x \in U \mid f(x) = \alpha\} = N_\alpha$.

The latter definition generalizes the concept of a KS-colouring: a KS-colouring is just a colouring compatible with the partition $(d-1, 1) \vdash d$.

5. A NEW SATURATED KOCHEN-SPECKER CONFIGURATION ON THREE QUBITS

The dimension of the Hilbert space of a single qubit is 2. The dimension of the Hilbert space \mathcal{H} of a triple of qubits is $d = 8$, $8 = 2 \times 2 \times 2$. In this section we are going to describe a new *saturated* Kochen-Specker configuration in \mathcal{H} consisting of $N = 64$ rays: $\mathbb{C}\psi_0, \mathbb{C}\psi_1, \dots, \mathbb{C}\psi_{N-1}$. Denote

$$[n] := \{0, 1, \dots, n-1\},$$

for $n \in \mathbb{Z}_{>0}$. One may write the representing vectors ψ_i as a line of eight numbers (the coordinates of a vector in a selected basis),

$$\psi_i = (\psi_i^{(0)}, \psi_i^{(1)}, \dots, \psi_i^{(d-1)}),$$

where $i \in [N]$. The inner product $\langle -, - \rangle$ on $\mathcal{H} = \mathbb{C}^d$ is then given by

$$\langle \psi_i, \psi_j \rangle = \sum_{k=0}^{d-1} \overline{\psi_i^{(k)}} \psi_j^{(k)},$$

where the bar denotes complex conjugation, $i, j \in [N]$.

The collection of vectors we are going to describe has an additional property:

$$\psi_i^{(k)} \in \{-1, 0, 1\},$$

for $i \in [N]$ and $k \in [d]$, so it makes sense to use a special notation. We write $\bar{1}$ instead of -1 and skip the spaces and commas between the coordinates of a vector: for example, $\psi = (1\bar{1}\bar{1}10000)$ is a vector $\psi = (1, -1, -1, 1, 0, 0, 0, 0)$.

Put:

$$\begin{aligned} \psi_0 &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}1), & \psi_1 &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_2 &:= (1\bar{1}\bar{1}\bar{1}1\bar{1}\bar{1}), \\ \psi_3 &:= (1\bar{1}\bar{1}\bar{1}11\bar{1}), & \psi_4 &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_5 &:= (1\bar{1}\bar{1}\bar{1}\bar{1}11), \\ \psi_6 &:= (1\bar{1}\bar{1}111\bar{1}), & \psi_7 &:= (1\bar{1}\bar{1}1111), & \psi_8 &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), \\ \psi_9 &:= (1\bar{1}\bar{1}\bar{1}\bar{1}1), & \psi_{10} &:= (1\bar{1}\bar{1}\bar{1}11\bar{1}), & \psi_{11} &:= (1\bar{1}\bar{1}\bar{1}111), \\ \psi_{12} &:= (1\bar{1}11\bar{1}\bar{1}\bar{1}), & \psi_{13} &:= (1\bar{1}11\bar{1}\bar{1}\bar{1}), & \psi_{14} &:= (1\bar{1}1111\bar{1}), \\ \psi_{15} &:= (1\bar{1}1111\bar{1}), & \psi_{16} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_{17} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), \\ \psi_{18} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_{19} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_{20} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), \\ \psi_{21} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}1), & \psi_{22} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_{23} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}1), \\ \psi_{24} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_{25} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}1), & \psi_{26} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), \\ \psi_{27} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}1), & \psi_{28} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_{29} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), \\ \psi_{30} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & \psi_{31} &:= (1\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}), & & \end{aligned} \tag{3}$$

and

$$\begin{aligned}
\psi_{32} &:= (1\bar{1}\bar{1}10000), & \psi_{33} &:= (1\bar{1}\bar{1}\bar{1}0000), & \psi_{34} &:= (1\bar{1}0000\bar{1}\bar{1}), \\
\psi_{35} &:= (1\bar{1}0000\bar{1}\bar{1}), & \psi_{36} &:= (1\bar{1}000011), & \psi_{37} &:= (1\bar{1}00001\bar{1}), \\
\psi_{38} &:= (1\bar{1}\bar{1}10000), & \psi_{39} &:= (1\bar{1}\bar{1}\bar{1}0000), & \psi_{40} &:= (11\bar{1}\bar{1}0000), \\
\psi_{41} &:= (11\bar{1}\bar{1}0000), & \psi_{42} &:= (110000\bar{1}\bar{1}), & \psi_{43} &:= (1100001\bar{1}), \\
\psi_{44} &:= (11000011), & \psi_{45} &:= (1100001\bar{1}), & \psi_{46} &:= (11110000), \\
\psi_{47} &:= (111\bar{1}0000), & \psi_{48} &:= (001\bar{1}\bar{1}100), & \psi_{49} &:= (0011\bar{1}100), \\
\psi_{50} &:= (00001\bar{1}\bar{1}\bar{1}), & \psi_{51} &:= (00001\bar{1}\bar{1}\bar{1}), & \psi_{52} &:= (00001\bar{1}11), \\
\psi_{53} &:= (00001\bar{1}\bar{1}\bar{1}), & \psi_{54} &:= (00111\bar{1}00), & \psi_{55} &:= (001\bar{1}\bar{1}100), \\
\psi_{56} &:= (001\bar{1}\bar{1}100), & \psi_{57} &:= (0011\bar{1}\bar{1}00), & \psi_{58} &:= (000011\bar{1}\bar{1}), \\
\psi_{59} &:= (000011\bar{1}\bar{1}), & \psi_{60} &:= (00001111), & \psi_{61} &:= (0000111\bar{1}), \\
\psi_{62} &:= (00111100), & \psi_{63} &:= (001\bar{1}1100).
\end{aligned} \tag{4}$$

Theorem 1. *The collection of rays (3), (4),*

$$\mathcal{M} := \{\mathbb{C}\psi_0, \mathbb{C}\psi_1, \dots, \mathbb{C}\psi_{63}\}$$

is a saturated Kochen-Specker configuration in $\mathcal{H} = \mathbb{C}^8$.

Proof. Denote $N = 64$, $d = 8$. One may use the following strategy to verify the Kochen-Specker statement. First, find all non-empty subsets $I \subseteq [N]$ such that $\mathbb{C}\psi_i \not\perp \mathbb{C}\psi_j$, for any $i, j \in I$. The function $f_I : \mathcal{M} \rightarrow \{0, 1\}$, defined as $f_I(\mathbb{C}\psi_i) = 1$, if $i \in I$, and $f_I(\mathbb{C}\psi_j) = 0$, if $j \in [N] \setminus I$, is a *candidate* for a Kochen-Specker colouring of \mathcal{M} . One needs to check, that for every I , the collection $\mathcal{M} \setminus \{\mathbb{C}\psi_i\}_{i \in I}$ always contains a tuple of d mutually orthogonal rays.

To generate the tuples of mutually orthogonal rays one can proceed as follows. Take $i_0 \in [N]$. Then find $i_1 \in [N]$ such that $i_1 > i_0$ and $\mathbb{C}\psi_{i_1} \perp \mathbb{C}\psi_{i_0}$. After that find $i_2 \in [N]$ such that $i_2 > i_1$ and $\mathbb{C}\psi_{i_2} \perp \mathbb{C}\psi_{i_1}, \mathbb{C}\psi_{i_0}$, and so on. Having a tuple $i_0 < i_1 < \dots < i_{s-1}$ of mutually orthogonal rays, $1 \leq s \leq d-1$, one needs to check that there always exists a $j \in [N] \setminus \{i_0, i_1, \dots, i_{s-1}\}$ such that $\mathbb{C}\psi_j \perp \mathbb{C}\psi_k$, for all $k \in \{i_0, i_1, \dots, i_{s-1}\}$. This establishes the saturation property.

For a realization of this method in C look at the supplementary file `saturated.c`. The supplementary file `vectors.txt` contains a sequence of $64 \times 8 = 512$ numbers a_0, a_1, \dots, a_{511} separated by a space, such that $\psi_i^{(k)} = a_{i*8+k}$, for $i \in [64]$, $k \in [8]$.

It is worth to point out how to check the Kochen-Specker property of \mathcal{M} *without* a computer. Our configuration \mathcal{M} is actually quite symmetric: for every element $x \in \mathcal{M}$ there exist exactly 31 elements $y \in \mathcal{M}$ such that $y \perp x$. Take the orthogonality graph $\Gamma_{\mathcal{M}}$ and look at 16 maximal cliques $C_m = \{\mathbb{C}\psi_i\}_{i \in K_m} \in \mathcal{P}_{\perp}^{(d)}(\mathcal{M})$, $m \in [16]$, $d = 8$, of

the shape:

$$\begin{aligned}
0 : \{0, 3, 13, 14, 20, 23, 25, 26\}, & \quad 1 : \{1, 2, 12, 15, 21, 22, 24, 27\}, \\
2 : \{0, 3, 13, 14, 21, 22, 24, 27\}, & \quad 3 : \{1, 2, 12, 15, 20, 23, 25, 26\}, \\
4 : \{33, 38, 40, 47, 51, 52, 58, 61\}, & \quad 5 : \{34, 37, 43, 44, 48, 55, 57, 62\}, \\
6 : \{33, 38, 43, 44, 48, 55, 58, 61\}, & \quad 7 : \{34, 37, 40, 47, 51, 52, 57, 62\}, \\
8 : \{4, 7, 9, 10, 16, 19, 29, 30\}, & \quad 9 : \{5, 6, 8, 11, 17, 18, 28, 31\}, \\
10 : \{4, 7, 9, 10, 17, 18, 28, 31\}, & \quad 11 : \{5, 6, 8, 11, 16, 19, 29, 30\}, \\
12 : \{32, 39, 41, 46, 50, 53, 59, 60\}, & \quad 13 : \{35, 36, 42, 45, 49, 54, 56, 63\}, \\
14 : \{32, 39, 42, 45, 49, 54, 59, 60\}, & \quad 15 : \{35, 36, 41, 46, 50, 53, 56, 63\},
\end{aligned}$$

where the notation $m : \{i_0, i_1, \dots, i_7\}$, $i_\alpha \in [64]$, $\alpha \in [8]$, $m \in [16]$, is a short way to write $K_m := \{i_0, i_1, \dots, i_7\}$.

Observe that $\{K_{4p+\alpha}\}_{p \in [4], \alpha=0,1}$ yields a *partition* of \mathcal{M} by maximal cliques. Therefore, if \mathcal{M} admits a KS-colouring then there must exist an anticlique $W \in \mathcal{P}_\chi^{(8)}(\mathcal{M})$ of size $\#W = 8$. It follows that to prove that \mathcal{M} is a Kochen-Specker type configuration it suffices to prove that an anticlique in \mathcal{M} of this size does not exist.

Note that the set $L_p := K_{4p} \cup K_{4p+1}$ can also be written as $L_p = K_{4p+2} \cup K_{4p+3}$, $p \in [4]$. It is straightforward to check that the sub-configurations $\mathcal{Q}_p := \{\mathbb{C}\psi_i\}_{i \in [L_p]}$, $p \in [4]$, have a property:

$$\forall x \in \mathcal{Q}_p : \#\{y \in \mathcal{Q}_p \mid y \neq x \& y \not\perp x\} = 4,$$

for $p \in [4]$. Since $\#\mathcal{Q}_p = 16$, we conclude that $\#\mathcal{P}_\chi^{(2)}(\mathcal{Q}_p) = (16 \times 4)/2 = 32$, for every $p \in [4]$. Furthermore, for the intersections $T_p^{(\alpha,\beta)} := K_{4p+\alpha} \cap K_{4p+2+\beta}$, where $\alpha, \beta \in [2]$, $p \in [4]$, we have $\#T_p^{(\alpha,\beta)} = 4$, so we obtain: $\mathcal{P}_\chi^{(2)}(\mathcal{Q}_p) = \{\{\mathbb{C}\psi_{i_0}, \mathbb{C}\psi_{i_1}\}\}_{(i_0, i_1) \in (T_p^{(0,0)} \times T_p^{(1,1)}) \cup (T_p^{(0,1)} \times T_p^{(1,0)})}$, for every $p \in [4]$. Next, it turns out that

$$\forall U \in \mathcal{P}_\chi^{(2)}(\mathcal{Q}_{2m}) \exists! V \in \mathcal{P}_\chi^{(2)}(\mathcal{Q}_{2m+1}) : U \cup V \in \mathcal{P}_\chi^{(4)}(\mathcal{Q}_{2m} \cup \mathcal{Q}_{2m+1}).$$

for every $m = 0, 1$. Hence, we are left with not so many variants: $\#\mathcal{P}_\chi^{(4)}(\mathcal{H}_m) = 32$, where $\mathcal{H}_m := \mathcal{Q}_{2m} \cup \mathcal{Q}_{2m+1}$, $m = 0, 1$. With a small amount of bookkeeping, one can check that

$$\forall U \in \mathcal{P}_\chi^{(4)}(\mathcal{H}_0) \forall V \in \mathcal{P}_\chi^{(4)}(\mathcal{H}_1) : U \cup V \notin \mathcal{P}_\chi^{(8)}(\mathcal{H}_0 \cup \mathcal{H}_1).$$

Therefore, the configuration $\mathcal{M} = \mathcal{H}_0 \cup \mathcal{H}_1$ does not have an anticlique of the required size, and we conclude that \mathcal{M} does not admit a KS-colouring. \square

Remark. Another way to prove that the configuration \mathcal{M} is of the Kochen-Specker type is discussed in the Appendix A. \diamond

Remark. The configuration \mathcal{M} does not admit a colouring compatible with the partition $(7, 1) \vdash 8$, but it admits other types of colourings.

Without going into details, one can mention that \mathcal{M} admits colourings compatible with the partitions $(6, 2) \vdash 8$ and $(4, 4) \vdash 8$, but not with the partition $(5, 3) \vdash 8$. An example of a colouring compatible with $(6, 2) \vdash 8$ is an indicator function $f = 1_{\mathcal{A}}$ of $\mathcal{A} = \{\mathbb{C}\psi_i\}_{i \in A}$, where

$$A := \{0, 1, 3, 4, 5, 7, 11, 15, 32, 33, 36, 37, 60, 61, 62, 63\},$$

and an example of a colouring compatible with $(4, 4) \vdash 8$ is an indicator function $f = 1_{\mathcal{B}}$ of $\mathcal{B} = \{\mathbb{C}\psi_i\}_{i \in B}$, where

$$B := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \\ 32, 33, 34, 35, 36, 37, 38, 39, 56, 57, 58, 59, 60, 61, 62, 63\}.$$

For every $U \in \mathcal{P}_{\perp}^{(8)}(\mathcal{M})$ holds: $\#U \cap \mathcal{A} = 2$, and $\#U \cap \mathcal{B} = 4$. \diamond

It is of interest to point out the following property of the configuration \mathcal{M} , $\#\mathcal{M} = 64$. If we look at all its maximal cliques with respect to the orthogonality relation \perp , then it turns out that the cardinality of an intersection $\#U \cap U'$, as $U, U' \in \mathcal{P}_{\perp}^{(8)}(\mathcal{M})$, $U \neq U'$, can be $0, 1, \dots, 6$, but is never 7. At the same time, the configuration \mathcal{M} is saturated, so we conclude:

$$\forall W \in \mathcal{P}_{\perp}^{(7)}(\mathcal{M}) \exists! U \in \mathcal{P}_{\perp}^{(8)}(\mathcal{M}) : U \supseteq W. \quad (5)$$

This property is similar to the property required in the definition of a *Steiner system* $S(t, k, n)$, $t = 7$, $k = 8$, $n = 64$, except that in the case of $S(64, 8, 7)$ arbitrary subsets of cardinality 7 are allowed, while in (5) one restricts to subsets of mutually orthogonal elements.

6. A NEW CRITICAL KOCHEN-SPECKER CONFIGURATION ON THREE QUBITS

In [18] one has constructed an example of a *saturated* Kochen-Specker configuration in 8-dimensional space. This configuration contains 120 rays and extends the Kochen-Specker example found by [27] (40 rays). It turns out that this 120 rays can be perceived as the rays represented by the 240 elements of the E_8 -root system (each ray is represented by a pair of roots v and $-v$).

In the previous section we have constructed a new *saturated* configuration formed by 64 rays in 8 dimensions. It turns out that the number of rays producing the Kochen-Specker property can be reduced at the expense of sacrificing the saturation property. In this section I describe a *critical* sub-configuration $\mathcal{N} \subseteq \mathcal{M}$ consisting of 36 rays. We keep the notation (3), (4), for the vectors $\psi_i \in \mathbb{C}^8$, $i \in [64]$.

Theorem 2. *The collection of rays*

$$\mathcal{N} := \{\mathbb{C}\psi_I\}_{i \in I},$$

where

$$I := \{2, 3, 4, 9, 12, 13, 14, 15, 16, 19, 21, 23, 24, 25, 26, 27, 29, 30, \\ 32, 33, 34, 37, 39, 40, 41, 43, 46, 48, 51, 52, 55, 58, 59, 60, 61, 62\} \quad (6)$$

is a critical Kochen-Specker configuration in $\mathcal{H} = \mathbb{C}^8$, $\#\mathcal{N} = 36$.

Proof. The fact that \mathcal{N} does not admit a Kochen-Specker colouring is checked on a personal computer in analogy with the configuration \mathcal{M} . One also needs to check that for every $i \in I$, the collection $\mathcal{N} \setminus \{\mathbb{C}\psi_i\}$ does admit a Kochen-Specker colouring.

For an implementation in C take the supplementary file `critical.c`. The sequence of numbers $i_0 < i_1 < \dots < i_{35}$ which form the set $I = \{i_\alpha\}_{\alpha=0}^{35}$ is written in a supplementary file `critseq.txt` using space as a separator. \square

Note that precisely half of the vectors ψ_i described by (6) representing the rays in \mathcal{N} have non-zero coordinates $\psi_i^{(k)} \in \{\pm 1\}$, $k \in [8]$, i.e. $i < 32$, while the other half comes from the $32 \leq i < 64$ part.

7. A NEW TROPICAL KOCHEN-SPECKER CONFIGURATION ON THREE QUBITS

Let \mathcal{M} be a collection of n rays in a d -dimensional Euclidean space \mathcal{H} . Recall that we have a notation $\mathcal{P}_\perp^{(k)}(\mathcal{M})$ for the collection of all mutually orthogonal k -tuples of rays in \mathcal{M} , $k \in \mathbb{Z}_{>0}$. If $k > d$ or $k > n$ then the set $\mathcal{P}_\perp^{(k)}(\mathcal{M})$ is empty. In analogy with this notation, denote $\mathcal{P}_\times^{(k)}(\mathcal{M})$ the collection of all k -tuples of mutually *non-orthogonal* rays $k \in \mathbb{Z}_{>0}$. Note, that in contrast with $\mathcal{P}_\perp^{(k)}(\mathcal{M})$, the class $\mathcal{P}_\times^{(k)}(\mathcal{M})$ does not need to be empty if $k > d$. At the same time, $\mathcal{P}_\perp^{(1)}(\mathcal{M})$ and $\mathcal{P}_\times^{(1)}(\mathcal{M})$ coincide being just a collection of singletons containing the elements of \mathcal{M} .

Consider a pair of functions $C_\mathcal{M}, \tilde{C}_\mathcal{M} : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{\geq 0}$,

$$C_\mathcal{M} : k \mapsto \#\mathcal{P}_\perp^{(k)}(\mathcal{M}), \quad \tilde{C}_\mathcal{M} : k \mapsto \#\mathcal{P}_\times^{(k)}(\mathcal{M}), \quad (7)$$

where k varies over $\mathbb{Z}_{>0}$. Note that if \mathcal{M} is just a single d -tuple of mutually orthogonal rays, then $C_\mathcal{M}(k)$, $k \in \mathbb{Z}_{>0}$, is just a binomial coefficient C_d^k .

Definition 12. *The pair of functions $(C_\mathcal{M}, \tilde{C}_\mathcal{M})$ associated with a collection of rays \mathcal{M} in \mathcal{H} is called a signature of \mathcal{M} .*

A signature (7) is a convenient concept which allows to formulate a *necessary* condition meant to distinguish a pair of configurations with non-isomorphic orthogonality graphs. If the signatures are different, then the orthogonality graphs cannot be isomorphic. At the same time, intuitively, if the the signatures of two configurations coincide, then there is a “high chance” that their orthogonality graphs are isomorphic, although some additional investigation is necessary to establish this isomorphism.

The signature of the critical Kochen-Specker configuration \mathcal{N} described by (6) is as follows:

$$(C_{\mathcal{N}}(1), C_{\mathcal{N}}(2), \dots, C_{\mathcal{N}}(8)) = (36, 346, 1224, 2063, 1776, 830, 204, 21),$$

$$(\tilde{C}_{\mathcal{N}}(1), \tilde{C}_{\mathcal{N}}(2), \tilde{C}_{\mathcal{N}}(3), \tilde{C}_{\mathcal{N}}(4)) = (36, 284, 536, 212),$$

and $C_{\mathcal{N}}(k) = 0$, if $k > 8$, and $\tilde{C}_{\mathcal{N}}(k) = 0$, if $k > 4$.

The signature of the saturated Kochen-Specker configuration \mathcal{M} described by (3), (4) is as follows:

$$(C_{\mathcal{M}}(1), \dots, C_{\mathcal{M}}(8)) = (64, 992, 5056, 11504, 13312, 8192, 2560, 320),$$

$$(\tilde{C}_{\mathcal{M}}(1), \tilde{C}_{\mathcal{M}}(2), \dots, \tilde{C}_{\mathcal{M}}(6)) = (64, 1024, 4864, 8512, 5632, 1536),$$

and $C_{\mathcal{M}}(k) = 0$, if $k > 8$, and $\tilde{C}_{\mathcal{M}}(k) = 0$, if $k > 6$.

It turns out that one can compute all tropical sub-configurations in \mathcal{M} and that all of them have the same signatures.

Theorem 3. *Let $\mathcal{M} = \{\mathbb{C}\psi_i\}_{i=0}^{63}$ be the saturated Kochen-Specker configuration in $\mathcal{H} = \mathbb{C}^d$, $d = 8$, described in (3), (4). The following holds:*

- 1) *The tropical dimension $\dim_d(\mathcal{M}) = 6$.*
- 2) *The number of tropical sub-configurations in \mathcal{M} of this tropical dimension is 32. Each of them contains 48 elements and their signatures coincide.*
- 3) *The critical Kochen-Specker configuration \mathcal{N} described above, $\#\mathcal{N} = 36$, is contained in exactly one of these tropical sub-configurations.*

Proof. The computation of the tropical dimension $\dim_d(\mathcal{M})$ can be done on a personal computer. To generate *all* tropical sub-configurations a program written in C needs about a day on a modest machine with a dual core Intel processor and 4GB RAM. The general strategy is based on several non-trivial observations.

Let us first introduce a new definition. Suppose we have a collection of maximal cliques $U_0, U_1, \dots, U_{q-1} \in \mathcal{P}_{\perp}^{(d)}(\mathcal{M})$. Let us say that this collection admits an *anticlique section* iff there exists an anticlique

$W \in \mathcal{P}_\chi^{(q)}(\mathcal{M})$ such that $\#W \cap U_i = 1$, for every $i = 0, 1, \dots, q-1$. In other words, there exists a function $f_W : \cup_{i=0}^{q-1} U_i \rightarrow \{0, 1\}$ such that: 1) for every $i \in [q]$ there exists a unique $x_i \in U_i$ such that $f_W(x_i) = 1$; 2) for every $i, j \in [q]$ holds: $f_W(x_j) \not\leq f_W(x_i)$.

Take a collection $U_0, U_1, \dots, U_{q-1} \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$ which admits an anticlique section f_W . Can the configuration of rays $\mathcal{T} := \cup_{i \in [q]} U_i$ be a Kochen-Specker configuration? The function f_W is a *candidate* for a KS-colouring. If \mathcal{T} does not admit KS-colourings then it must be possible to find $V \in \mathcal{P}_\perp^{(d)}(\mathcal{T} \setminus f_W^{-1}(\{1\}))$. Let $f_W^{-1}(\{1\}) = \{x_0, x_1, \dots, x_{q-1}\}$, $x_i \in U_i$, $f(x_i) = 1$, $i \in [q]$. Since for any $y \in \mathcal{T} \setminus f_W^{-1}(\{1\})$ there exists $i \in [q]$ such that $x_i \perp y$, we conclude that the collection of $q+1$ maximal cliques $\{U_0, U_1, \dots, U_{q-1}, V\}$ does *not* admit an anticlique section. We also observe that

$$\#(V \cup (\cup_{i \in [q]} U_i)) = \# \cup_{i \in [q]} U_i \leq qd.$$

It is rather straightforward to generate all maximal cliques $U \in \mathcal{P}_\perp^{(d)}(\mathcal{M})$. The total number turns out to be $n = 320$, so we denote them as U_i , $i \in [n]$. A further computation yields the following fact. For any collection $i_0 < i_1 < \dots < i_{q-1}$, $i_\alpha \in [n]$, $\alpha \in [q]$, if $q \leq 5$ then the collection $\{U_{i_\alpha}\}_{\alpha \in [q]}$ admits an anticlique section. Therefore we conclude that $\dim_d(\mathcal{M}) \geq 5$.

On the other hand (and this is the longest part of the computation), there exist 6-tuples $\{i_0 < i_1 < \dots < i_5\}$ such that $\{U_{i_\alpha}\}_{\alpha \in [6]}$ does not admit an anticlique section. The number of variants of such tuples is $N = 308992$ and we denote the corresponding variants as $\{i_0^{(\beta)} < i_1^{(\beta)} < \dots < i_5^{(\beta)}\}$, $\beta \in [N]$. The corresponding computations can be found in the supplementary file `antisect.c`.

It turns out that for every $\beta \in [N]$ holds:

$$\# \cup_{\alpha \in [6]} U_{i_\alpha^{(\beta)}} = 48.$$

The dimension of space in our case is $d = 8$, so, since $48 = 6 \times 8$, we immediately conclude that every collection $\{U_{i_\alpha^{(\beta)}}\}_{\alpha \in [6]}$ is a collection of *mutually disjoint* maximal cliques, $\beta \in [N]$. Intuitively, one may perceive this observation as an effect of “*repulsion*” of cliques: the corresponding configuration tries to be as large as possible. Since $48 > 40 = 5 \times 8$, we exclude the possibility of $\dim_d(\mathcal{M}) = 5$.

The unions $\cup_{\alpha \in [6]} U_{i_\alpha^{(\beta)}}$ are Kochen-Specker configurations, since any KS-colouring of such a union would induce an anticlique section of $\{U_{i_\alpha^{(\beta)}}\}_{\alpha \in [6]}$, $\beta \in [N]$. So our conclusion is as follows:

$$\dim_d(\mathcal{M}) = 6.$$

If we compute the image of the map $[N] \ni \beta \mapsto \cup_{\alpha \in [6]} U_{i_\alpha}^{(\beta)} \in \mathcal{P}(\mathcal{M})$, then it turns out that it contains only 32 different variants. It is straightforward to check that their signatures coincide.

Write the 32 tropical configurations mentioned as follows:

$$\mathcal{T}_m = \{\mathbb{C}\psi_i\}_{i \in J_m},$$

where

$$J_m = \{j_m^{(0)} < j_m^{(1)} < \dots < j_m^{(47)}\},$$

for $m \in [32]$. The supplementary file `tropseq.txt` contains a sequence of $32 \times 48 = 1536$ numbers $b_0, b_1, \dots, b_{1535}$, separated by a space, such that $j_m^{(s)} = b_{m*48+s}$, $m \in [32]$, $s \in [48]$.

The tropical sub-configuration of \mathcal{M} containing \mathcal{N} is \mathcal{T}_0 . The corresponding set J_0 is as follows:

$$\begin{aligned} J_0 = \{0, 1, 2, 3, 4, 7, 9, 10, 12, 13, 14, 15, 16, 19, 20, 21, \\ 22, 23, 24, 25, 26, 27, 29, 30, 32, 33, 34, 37, 38, 39, 40, 41, \\ 43, 44, 46, 47, 48, 50, 51, 52, 53, 55, 57, 58, 59, 60, 61, 62\}. \end{aligned} \quad (8)$$

We have: $\mathcal{N} \subseteq \mathcal{T}_0 \subseteq \mathcal{M}$ and $\#\mathcal{T}_0 = 48$. The computations are implemented in the supplementary file `tropical.c`. \square

The signature of the configuration $\mathcal{T} = \mathcal{T}_0$ described by (8) in the proof of the theorem is of the shape:

$$\begin{aligned} (C_{\mathcal{T}}(1), C_{\mathcal{T}}(2), \dots, C_{\mathcal{T}}(8)) &= (48, 600, 2752, 6096, 7008, 4304, 1344, 168), \\ (\tilde{C}_{\mathcal{T}}(1), \tilde{C}_{\mathcal{T}}(2), \dots, \tilde{C}_{\mathcal{T}}(5)) &= (48, 528, 1536, 1312, 384), \end{aligned}$$

and $C_{\mathcal{T}}(k) = 0$, if $k > 8$, and $\tilde{C}_{\mathcal{T}}(k) = 0$, if $k > 5$.

8. THIRTY-SIX.

There is another known example of a critical Kochen-Specker type configuration due to [27]. It so happens that it also contains 36 vectors, just like the configuration \mathcal{N} described by (6). Are these configurations equivalent or not? In other words, are the corresponding orthogonality graphs isomorphic or not?

The critical configuration discovered in [27] can be described as follows. Consider first a configuration

$$\mathcal{T}' := \{\mathbb{C}\varphi_i\}_{i \in [40]},$$

represented by 40 vectors of the shape:

$$\begin{aligned}
\varphi_0 &:= (10000000), & \varphi_1 &:= (01000000), & \varphi_2 &:= (00100000), \\
\varphi_3 &:= (00010000), & \varphi_4 &:= (00001000), & \varphi_5 &:= (00000100), \\
\varphi_6 &:= (00000010), & \varphi_7 &:= (00000001), & \varphi_8 &:= (11110000), \\
\varphi_9 &:= (11\bar{1}\bar{1}0000), & \varphi_{10} &:= (1\bar{1}\bar{1}10000), & \varphi_{11} &:= (1\bar{1}\bar{1}10000), \\
\varphi_{12} &:= (00001111), & \varphi_{13} &:= (000011\bar{1}\bar{1}), & \varphi_{14} &:= (00001\bar{1}\bar{1}\bar{1}), \\
\varphi_{15} &:= (00001\bar{1}\bar{1}1), & \varphi_{16} &:= (11001100), & \varphi_{17} &:= (1100\bar{1}\bar{1}00), \\
\varphi_{18} &:= (1\bar{1}001\bar{1}00), & \varphi_{19} &:= (1\bar{1}00\bar{1}100), & \varphi_{20} &:= (00110011), \\
\varphi_{21} &:= (001100\bar{1}\bar{1}), & \varphi_{22} &:= (001\bar{1}001\bar{1}), & \varphi_{23} &:= (001\bar{1}00\bar{1}1), \\
\varphi_{24} &:= (10101010), & \varphi_{25} &:= (1010\bar{1}0\bar{1}0), & \varphi_{26} &:= (10\bar{1}010\bar{1}0), \\
\varphi_{27} &:= (10\bar{1}0\bar{1}010), & \varphi_{28} &:= (01010101), & \varphi_{29} &:= (01010\bar{1}0\bar{1}), \\
\varphi_{30} &:= (010\bar{1}010\bar{1}), & \varphi_{31} &:= (010\bar{1}0\bar{1}01), & \varphi_{32} &:= (100101\bar{1}0), \\
\varphi_{33} &:= (100\bar{1}0110), & \varphi_{34} &:= (10010\bar{1}10), & \varphi_{35} &:= (100\bar{1}0\bar{1}10), \\
\varphi_{36} &:= (0110\bar{1}001), & \varphi_{37} &:= (01\bar{1}01001), & \varphi_{38} &:= (0\bar{1}101001), \\
\varphi_{39} &:= (0\bar{1}\bar{1}0\bar{1}001).
\end{aligned}$$

After that construct another configuration by excluding four vectors:

$$\mathcal{N}' := \{\mathbb{C}\varphi_i\}_{i \in [40] \setminus \{0, 12, 22, 31\}}.$$

Proposition 1. *The configuration \mathcal{N}' is a critical Kochen-Specker configuration on three qubits.*

Remark. There is a typo in the original paper [27]: by accident, the authors exclude the vector $\varphi_{27} = (10\bar{1}0\bar{1}010)$ instead of $\varphi_{31} = (010\bar{1}0\bar{1}01)$. \diamond .

The configuration \mathcal{N}' is an analogue of our configuration \mathcal{N} , and \mathcal{T}' is an analogue of the tropical configuration \mathcal{T} . A *saturated* extension \mathcal{M}' of \mathcal{T}' has been constructed in [18]. It turns out that $\#\mathcal{M}' = 120$ and that the rays of \mathcal{M}' can be represented by the vectors of the E_8 root system (this is an observation related to the question about the symmetry of \mathcal{N}' stated in [27]). We have:

$$\mathcal{N}' \subseteq \mathcal{T}' \subseteq \mathcal{M}'.$$

The configurations \mathcal{N} and \mathcal{N}' have the same cardinalities,

$$\#\mathcal{N}' = 36, \quad \#\mathcal{N} = 36,$$

but their signatures are different:

$$\begin{aligned}
(C_{\mathcal{N}'}(1), C_{\mathcal{N}'}(2), \dots, C_{\mathcal{N}'}(8)) &= (36, 374, 1384, 1991, 1120, 416, 96, 11), \\
(\tilde{C}_{\mathcal{N}'}(1), \tilde{C}_{\mathcal{N}'}(2), \tilde{C}_{\mathcal{N}'}(3), \tilde{C}_{\mathcal{N}'}(4)) &= (36, 256, 448, 192),
\end{aligned}$$

and $C_{\mathcal{N}'}(k) = 0$, if $k > 8$, and $\tilde{C}_{\mathcal{N}'}(k) = 0$, if $k > 4$. Therefore the configuration \mathcal{N} can not be isomorphic to \mathcal{N}' .

The signature of the configuration \mathcal{T}' is of the shape:

$$(C_{\mathcal{T}'}(1), C_{\mathcal{T}'}(2), \dots, C_{\mathcal{T}'}(8)) = (40, 460, 1880, 2990, 1880, 780, 200, 25),$$

$$(\tilde{C}_{\mathcal{T}'}(1), \tilde{C}_{\mathcal{T}'}(2), \tilde{C}_{\mathcal{T}'}(3), \tilde{C}_{\mathcal{T}'}(4)) = (40, 320, 640, 320),$$

and $C_{\mathcal{T}'}(k) = 0$, if $k > 8$, and $\tilde{C}_{\mathcal{T}'}(k) = 0$, if $k > 4$.

Let us also give (for the reference purposes) the signature of the E_8 configuration \mathcal{M}' :

$$(C_{\mathcal{M}'}(1), C_{\mathcal{M}'}(2), \dots, C_{\mathcal{M}'}(8)) =$$

$$= (120, 3780, 37800, 122850, 113400, 56700, 16200, 2025),$$

$$(\tilde{C}_{\mathcal{M}'}(1), \tilde{C}_{\mathcal{M}'}(2), \dots, \tilde{C}_{\mathcal{M}'}(8)) =$$

$$= (120, 3360, 31360, 120960, 241920, 241920, 103680, 8640),$$

and $C_{\mathcal{M}'}(k) = 0$ and $\tilde{C}_{\mathcal{M}'}(k) = 0$, if $k > 8$.

Note that it is of interest to search for multi-qubit generalizations of these configurations [23, 25]. In [20] one can find an infinite family of Kochen-Specker type configurations generalizing \mathcal{T}' on any number of qubits $n = 4m - 1$, $m \geq 1$.

9. CLIQUE PARTITIONS

In the previous section we have found all tropical sub-configurations in the saturated Kochen-Specker configuration $\mathcal{M} = \{\mathbb{C}\psi_i\}_{i=0}^{63}$ in $\mathcal{H} = \mathbb{C}^8$. We denote them as \mathcal{T}_i , $i = 0, 1, \dots, 31$. We have $\#\mathcal{T}_i = 48$, $i \in [32]$, and we know that the signatures $(C_{\mathcal{T}_i}, \tilde{C}_{\mathcal{T}_i})$ are the same for each $i \in [32]$. Since $48 = 6 \times 8 = \dim_{\mathbb{A}}(\mathcal{M}) \times \dim_{\mathbb{C}} \mathcal{H}$, it follows that every \mathcal{T}_i , $i \in [32]$, can be *partitioned* by maximal cliques $\mathcal{T}_i = U_0^{(i)} \cup U_1^{(i)} \cup \dots \cup U_5^{(i)}$, where $U_{\alpha}^{(i)} \in \mathcal{P}_{\perp}^{(8)}(\mathcal{N})$, and $U_{\alpha}^{(i)} \cap U_{\beta}^{(i)} = \emptyset$, if $\alpha \neq \beta$, for $\alpha, \beta \in [6]$, $i \in [32]$.

It is natural to investigate if there exist partitions by maximal cliques of the saturated configuration. It turns out that there are quite many, but there is a distinguished subset among them “induced” by the tropical sub-configurations.

Proposition 2. *Let $\mathcal{M} = \{\mathbb{C}\psi_i\}_{i \in [64]}$ be the saturated Kochen-Specker configuration as above, and let $\{\mathcal{T}_i\}_{i \in [32]}$ be the collection of all its tropical sub-configurations, $\#\mathcal{T}_i = 48$, $i \in [32]$. There exists a collection $\{B_{\alpha}\}_{\alpha \in [32]}$ of maximal cliques $B_{\alpha} \in \mathcal{P}_{\perp}^{(8)}(\mathcal{M})$, $\alpha \in [32]$, with the following property: for every $i \in [32]$, there exists a clique partition*

$$\mathcal{M} = B_{\alpha_0} \cup B_{\alpha_1} \cup \dots \cup B_{\alpha_7},$$

where $\alpha_0, \alpha_1, \dots, \alpha_7 \in [32]$, $\alpha_0 < \alpha_1 < \dots < \alpha_7$, such that the tropical sub-configuration $\mathcal{T}_i \subseteq \mathcal{M}$ can be represented in the form:

$$\mathcal{T}_i = B_{\alpha_{i_0}} \cup B_{\alpha_{i_1}} \cup \dots \cup B_{\alpha_{i_5}},$$

where $i_0, i_1, \dots, i_5 \in [8]$, $i_0 < i_1 < \dots < i_5$.

Proof. Write:

$$B_\alpha = \{\mathbb{C}\psi_i\}_{i \in K_\alpha},$$

where

$$K_\alpha = \{k_\alpha^{(0)} < k_\alpha^{(1)} < \dots < k_\alpha^{(7)}\},$$

for $\alpha \in [32]$. The supplementary file `baseseq.txt` contains a sequence of $32 \times 8 = 256$ numbers c_0, c_1, \dots, c_{255} separated by a space, such that $k_\alpha^{(s)} = c_{\alpha*8+s}$, for $\alpha \in [32]$, $s \in [8]$.

A straightforward computation (see the supplementary file `bases.c`) yields the following facts. Every tropical sub-configuration \mathcal{T}_i , $i \in [32]$, can be represented in 48 different ways in the form $\mathcal{T}_i = B_{\beta_0} \cup B_{\beta_1} \cup \dots \cup B_{\beta_5}$, where $\beta_0 < \beta_1 < \dots < \beta_5$, $\beta_j \in [32]$, $j \in [6]$. There are 576 clique partitions $\mathcal{M} = B_{\alpha_0} \cup B_{\alpha_1} \cup \dots \cup B_{\alpha_7}$, where $\alpha_0 < \alpha_1 < \dots < \alpha_7$, $\alpha_i \in [32]$, $i \in [8]$. Every partition of \mathcal{T}_i extends in two different ways to a partition of \mathcal{M} by the cliques $\{B_\alpha\}_{\alpha \in [32]}$ and this way there are 96 such partitions in total corresponding to a given tropical sub-configuration. \square

We have a collection of maximal cliques $\mathcal{W} := \{B_i\}_{i \in [32]} \subseteq \mathcal{P}_\perp^{(8)}(\mathcal{M})$ associated with the tropical sub-configurations $\{\mathcal{T}_i\}_{i \in [32]}$. Let $I \subset [32]$ be a non-empty subset such that $\{B_i\}_{i \in I}$ are mutually disjoint. Denote the collection of all such subsets $\mathcal{D}(\mathcal{W})$, and put $\mathcal{M}_I^\mathcal{W} := \cup_{i \in I} B_i$ for $\{B_i\}_{i \in I} \in \mathcal{D}(\mathcal{W})$. Define a pair of functions

$$F_\mathcal{W} : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}, \quad s \mapsto \#\{I \in \mathcal{D}(\mathcal{W}) \mid \#I = s\},$$

and

$$G_\mathcal{W} : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}, \quad s \mapsto \#\{I \in \mathcal{D}(\mathcal{W}) \mid \#I = s \text{ and } \mathcal{M}_I^\mathcal{W} \text{ is saturated}\},$$

The corresponding computation yields:

$$\begin{aligned} (F_\mathcal{W}(1), F_\mathcal{W}(2), \dots, F_\mathcal{W}(8)) &= \\ &= (32, 400, 2496, 8304, 14592, 12672, 4608, 576), \end{aligned}$$

$$(G_\mathcal{W}(1), G_\mathcal{W}(2), \dots, G_\mathcal{W}(8)) = (32, 192, 480, 512, 192, 768, 0, 576),$$

and $F_\mathcal{W}(k) = 0$ and $G_\mathcal{W}(k) = 0$, if $k > 8$. It follows that we obtain a plenty of *saturated* sub-configurations in \mathcal{M} (and by that a family of orthoalgebras [19]). On the other hand, none of them happens to be a Kochen-Specker configuration, except for the whole collection \mathcal{M}

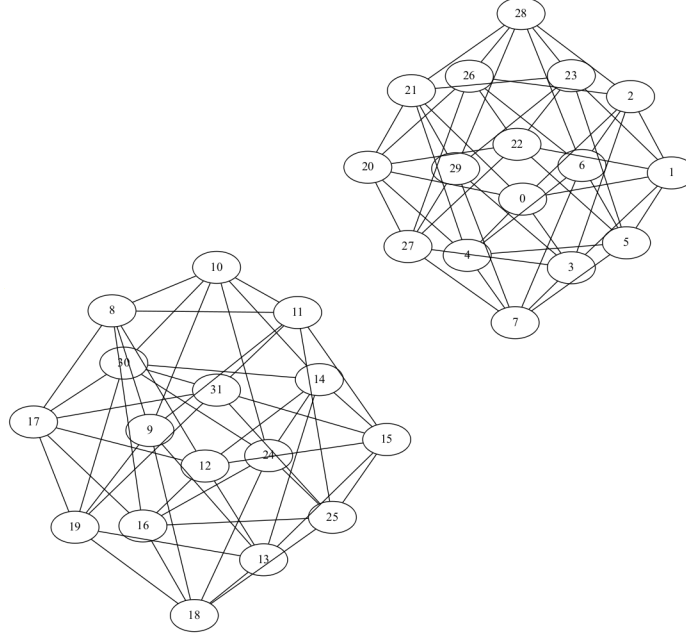


FIGURE 1. A pair of 4×4 rook's graphs corresponding to the bases described by `baseseq.txt`.

itself. The corresponding computations are given in the supplementary file `bases.c`.

It is of interest to point out the following special property of the collection of bases $\mathcal{W} := \{B_i\}_{i \in [32]}$. If we look at the cardinalities of intersections $\#B_\alpha \cap B_\beta$, $\alpha \neq \beta$, $\alpha, \beta \in [32]$, then it turns out that there are only two possibilities: either the two bases are disjoint, or the cardinality of intersection is 4 (half of the dimension of space). It follows that there is a canonically defined undirected graph $\Gamma = (V, E)$, with vertices $V = [32]$ and the edges E , $\{\alpha, \beta\} \in E$ iff $\#B_\alpha \cap B_\beta = 4$, where $\alpha, \beta \in [32]$, $\alpha \neq \beta$. It turns out that this graph falls apart into two isomorphic connected components (see the figure) each being a strongly perfect graph of type $(16, 6, 2, 2)$. There are *two* possibilities in this case (isomorphism classes): the 4×4 -rook's graph and the Shrikhande graph. It turns out that our graphs are the 4×4 -rook's graphs what can be established by looking at the neighbourhood of a vertex: in Shrikhande graph the neighbourhood is a cycle of length 6 and in the 4×4 -rook's case it is a pair of triangles.

10. CONCLUSION AND DISCUSSION

The main result of the present paper is a new *saturated* configuration of 64 rays containing a new *critical* Kochen-Specker configuration of 36 rays on three qubits (8-dimensional space). It turns out that this saturated configuration has quite nice symmetry properties which can be studied in terms of *tropical* sub-configurations discussed in this paper.

I would like to focus once more on several conceptual issues mentioned in the main text and make some informal remarks. First of all, it is quite natural to expect that the definition of a Kochen-Specker colouring (the latter is a certain function on a configuration with values 0 and 1) should have a “multicoloured” generalization. Intuitively, one may think that these colours can be perceived as elements of some finite group or a similar algebraic structure. From the perspective of graph theory such a generalization appears definitely possible, but the problem is that it is not immediately clear how to construct a generalization which would still be of interest in quantum mechanics, but not just in pure mathematics. The approach suggested in the present paper is based on the concept of an *entropy* of a *saturated* configuration. The important step is the interpretation of a configuration of rays which admits a Kochen-Specker colouring as a configuration with entropy equal to zero. A natural development of this idea leads to a physically reasonable multi-coloured generalization of Kochen-Specker theory and is captured by a definition of a *rational* Kochen-Specker colouring.

I would also like to stress once again the importance of saturated configurations. At first sight, an extension of a given critical Kochen-Specker configuration to a saturated one is not really necessary since it only complicates things, i.e. adds new measuring devices to an experimental set-up as if there is not enough trouble with decoherence of quantum states. Nonetheless a saturated configuration provides a natural environment where a critical configuration “lives”. A saturated Kochen-Specker configuration is a rather special object with many symmetries and in general one would expect many isomorphic copies of a given critical configuration inside it. There can be different isomorphism classes of critical configurations. In a sense, a saturated configuration (or its orthogonality graph) yields upto a certain point a discrete analogue of the space of pure states of a quantum system. More generally one can try to mimic quantum mechanics on finite undirected graphs with the property that every maximal clique has the same size.

The objects that exist inside a saturated Kochen-Specker configuration \mathcal{M} (e.g. other saturated or non-KS-colourable sub-configurations

\mathcal{N}) can “move”: if we take an automorphism of the orthogonality graph of \mathcal{M} , then $\mathcal{N} \subseteq \mathcal{M}$ does not need to stay fixed but it is transferred to an isomorphic copy of itself \mathcal{N}_1 . A sub-configuration \mathcal{N} in \mathcal{M} has a *signature* (the numbers of cliques and anti-cliques in the orthogonality graph). The most important parameter present in the signature is the *capacity* of a configuration \mathcal{N} : the number of maximal cliques. Intuitively, the higher is the capacity, the higher is the chance that a configuration does not admit a KS-colouring. It is tempting to term the number of elements in \mathcal{N} as *inductivity* of a configuration. A pair of sub-configurations \mathcal{N}_1 and \mathcal{N}_2 in \mathcal{M} can “interact”: if the capacity of \mathcal{N}_1 is n_1 , and the capacity of \mathcal{N}_2 is n_2 , then the capacity n of $\mathcal{N}_1 \cup \mathcal{N}_2$ is at least $n_1 + n_2$, but it can be $n > n_1 + n_2$. The maximal value of n corresponds to a “resonance effect”, etc.

In this paper we have encountered a collection of parameters (dimensions) $\dim_{k+1}(\mathcal{M})$, $k \in [d]$, which describe a saturated Kochen-Specker configuration \mathcal{M} (in particular, the critical dimension $\dim_1(\mathcal{M})$ and the tropical dimension $\dim_d(\mathcal{M})$). Another important characteristic is the *signature* of a configuration. These are not the only parameters that are of interest in practice. For example, one may look for the largest saturated sub-configuration $\mathcal{W} \subseteq \mathcal{M}$ which still admits a KS-colouring. This yields another parameter $d(\mathcal{M}) := \#\mathcal{W}$ describing \mathcal{M} . It is natural to term a saturated Kochen-Specker configuration *irreducible* iff it does not contain a saturated Kochen-Specker sub-configuration of smaller inductivity. Once one uses a multi-coloured version of the Kochen-Specker theory (for example, one is counting the colourings compatible with a partition of the dimension of space into a sum of positive integers), one may introduce similar dimensions and signatures by analogy.

APPENDIX A

It is of interest to point out a link between the configurations discussed in the present paper and the recent work of M. Waegell and P. K. Aravind [26], where they make an important observation about the algebraic nature of a certain class of proofs of the Kochen-Specker theorem. Consider the matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix I is the 2×2 identity matrix, and X , Y , and Z , are related to the Pauli matrices as follows: $\sigma_1 = X$, $\sigma_2 = -iY$, $\sigma_3 = Z$. Write

$$X^{(0)} := I, \quad X^{(1)} := X, \quad X^{(2)} := Y, \quad X^{(3)} := Z,$$

and put

$$A^{(\alpha)} := X^{(\alpha_0)} \otimes X^{(\alpha_1)} \otimes X^{(\alpha_2)}, \quad (9)$$

where $\alpha \in [64] = \{0, 1, \dots, 63\}$, $\alpha = \alpha_0 + 4\alpha_1 + 16\alpha_2$, for $\alpha_0, \alpha_1, \alpha_2 \in [4] = \{0, 1, 2, 3\}$. A rather special property of the collection of matrices $\{A^{(\alpha)}\}_{\alpha \in [64]}$ is that there exist quite many variants to choose tuples $\mu_0 < \mu_1 < \dots < \mu_{s-1}$, where $\mu_i \in [64]$, $i \in [s]$, $s = 3, 4, 7$, such that

$$[A^{(\mu_i)}, A^{(\mu_j)}] = 0, \quad \prod_{k \in [s]} A^{(\mu_k)} = \pm 1, \quad (10)$$

where $i, j \in [s]$, and $[-, -]$ denotes the commutator, the order of the factors in the product does not matter since the matrices commute.

Suppose we have a hyper-graph with n vertices labeled by $A^{(\nu_i)}$, $\nu_i \in [64]$, $\nu_i \neq \nu_j$, if $i \neq j$, $i, j \in [n]$, and with hyper-edges of cardinalities 3 or 4. Assume that the labels of the vertices in every hyper-edge correspond to the tuples of the shape (10). A hyper-edge is termed *negative* if the product of the labels of all its vertices is -1 , and it is termed *positive* if this product is $+1$. In [26] it is pointed out that every time we have a hyper-graph like this with an *odd* number of negative hyper-edges, then if we have a property that every vertex is contained in an *even* number of hyper-edges, then this yields immediately a proof of the Kochen-Specker theorem. A complete characterization of this class of hyper-graphs is an open mathematical problem. In the paper mentioned the authors provide explicitly a series of examples of the hyper-graphs they have found.

We observe now that the saturated Kochen-Specker configuration \mathcal{M} (64 rays) obtained the present paper admits a proof from the class [26]. At the same time this *configuration of rays* does not underlie any of the proofs mentioned in [26]. We also notice that a straightforward search on a computer (i.e. without using any symmetries of the Pauli group) for a saturated Kochen-Specker configuration with $N = 2^6$ rays would require going through $C_{135}^8 = 2214919483920$ variants (the lower index 135 in the binomial coefficient C_{135}^8 is the total number of all tuples of the type (10) of maximal possible length $s = 7$, and the upper index 8 should be perceived as N/d , where $d = 2^3$ is the dimension of the Euclidian space of three qubits).

It is a common convention to drop the symbol \otimes in the tensor product (9). If we use the notation I, X, Y, Z , then we write, for example, just IXX in place of $X^{(0)} \otimes X^{(1)} \otimes X^{(1)}$, YZX in place of $X^{(2)} \otimes X^{(3)} \otimes X^{(1)}$,

etc. Consider a list:

0 :	XII	IXI	XXI	IIZ	XIZ	IXZ	XXZ
1 :	XII	IXX	XXX	IYY	XYY	IZZ	XZZ
2 :	IXI	ZIX	ZXX	YIY	YXY	XIZ	XXZ
3 :	XXI	YYX	ZZX	ZYY	YZY	XIZ	IXZ
4 :	ZXI	YYI	XZI	IIZ	ZXZ	YYZ	XZZ
5 :	ZXI	ZIX	IXX	XYY	YZY	YYZ	XZZ
6 :	YYI	XXX	ZZX	YIY	IYY	ZXZ	XZZ
7 :	XZI	ZXX	YYX	YXY	ZYY	XIZ	IZZ

Every line of this list contains a set of 7 mutually commuting operators. Each time there is a set of 8 mutually orthogonal one-dimensional joint eigenspaces (rays) associated to it. These sets of rays corresponding to different lines of the list are mutually disjoint, and their union yields a set of 64 rays. This is precisely the *saturated* KS configuration \mathcal{M} introduced in the paper, if we assume that the matrices $A^{(\alpha)}$, $\alpha \in [64]$, act on $x = (x_0, x_1, \dots, x_7)$ as follows: $A^{(\alpha)}.x = y$, $y = (y_0, y_1, \dots, y_7)$:

$$y_{i_0+2i_1+4i_2} = \sum_{j_0, j_1, j_2=0,1} X_{i_0, j_0}^{(\alpha_0)} X_{i_1, j_1}^{(\alpha_1)} X_{i_2, j_2}^{(\alpha_2)} x_{j_0+2j_1+4j_2},$$

for $i_0, i_1, i_2 = 0, 1$, and $\alpha_0 + 4\alpha_1 + 16\alpha_2 = \alpha$, where $\alpha_0, \alpha_1, \alpha_2 \in [4]$, and $X_{k,l}^{(m)}$ denotes the element of the matrix $X^{(m)}$, $m \in [4]$, standing in the $(k+1)$ -th row, $k \in [2]$, and the $(l+1)$ -th column, $l \in [2]$.

Let us point out a proof that the configuration of rays \mathcal{M} is of the Kochen-Specker type. The collection of matrices present in the list above contains 26 elements:

$$\begin{aligned} & XII, IXI, XXI, ZXI, YYI, XZI, ZIX, IXX, XXX, \\ & ZXX, YYX, ZZX, YIY, YXY, IYY, XYY, ZYY, YZY, \\ & IIZ, XIZ, IXZ, XXZ, ZXZ, YYZ, IZZ, XZZ. \end{aligned} \quad (11)$$

Consider the following subsets of this collection:

0 :	IXI	ZIX	YXY	XIZ
1 :	IXI	ZXX	YIY	XIZ
2 :	ZXI	YYI	IIZ	XZZ
3 :	ZXI	XZI	IIZ	YYZ
4 :	YYI	XXX	YIY	XZZ
5 :	XZI	ZXX	YXY	IZZ
6 :	ZIX	IXX	YYZ	XZZ
*7 :	IXX	XXX	IZZ	XZZ

There are 15 matrices present in the list:

$$\begin{aligned} &IXI, ZXI, YYI, XZI, ZIX, IXX, XXX, \\ &ZXX, YIY, YXY, IIZ, XIZ, YYZ, IZZ, XZZ. \end{aligned}$$

Construct a hyper-graph on 15 vertices labeled with these matrices. The 4-tuples in the lines of the list above define 8 hyper-edges: the first seven are *negative* and the hyper-edge corresponding to the last line marked with a star is *positive*. The matrix IZZ occurs 4 times in a hyper-edge, and all other matrices occur twice in a hyper-edge. The underlying configuration of rays is, therefore, of the Kochen-Specker type.

The hyper-graph mentioned contains just one positive hyper-edge and all other hyper-edges are negative. All hyper-edges are of cardinality 4. This is not the only hyper-graph of this shape which corresponds to a proof that \mathcal{M} of the Kochen-Specker type, but, perhaps, the most simple one. The other hyper-graphs with a single positive hyper-edge can be generated on a personal computer as follows. There are 24 negative hyper-edges of size 4, and 54 positive hyper-edges of size 4 which can be constructed from the 26 matrices (11). 2^{24} is not a “huge” number compared to 2^{54} . Look at all hyper-graphs built only from an *odd* number of negative hyper-edges of size 4. Count for each vertex the number of hyper-edges containing it, and keep only those hyper-graphs which contain exactly 4 vertices corresponding to an odd number of hyper-edges. Then check if this 4-tuple of vertices forms a *positive* hyper-edge of size 4. If this is the case, we obtain a proof that the configuration is of the Kochen-Specker type. Since every hyper-graph contains just one *positive* hyper-edge, it is natural to classify these hyper-graphs by the tuples of numbers $\{n_k\}_{k \in [24]}$, where n_k is the number of vertices contained in exactly k *negative* hyper-edges. The computation yields 33 variants of $\{n_k\}_{k \in [24]}$. In principle, it would be of interest to describe all isomorphism classes of KS proofs for \mathcal{M} corresponding to any number of positive hyper-edges.

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